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Robust Momentum Management and Attitude Control System for the Space Station

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A game theoretic controller is synthesized for momentum management and attitude control of the Space Station in the presence of uncertainties in the moments of inertia. Full state information is assumed since attitude and attitude rates are assumed to be very accurately measured. By an input-output decomposition of the uncertainty in the system matrices, the parameter uncertainties in the dynamic system are represented as an unknown gain associated with an internal feedback loop (IFL). The input and output matrices associated with the IFL form directions through which the uncertain parameters affect system response. If the quadratic form of the IFL output augments the cost criterion, then enhanced parameter robustness is anticipated. By considering the input and the input disturbance from the IFL as two noncooperative players, a linear-quadratic differential game is constructed. The solution in the form of a linear controller is used for synthesis. Inclusion of the external disturbance torques results in a dynamic feedback controller that consists of conventional proportional-integral-derivative control and cyclic disturbance rejection filters. It is shown that the game theoretic design allows large variations in the inertias in directions of importance.

I. Introduction

A GAME theoretic controller developed in Refs. 1 and 22 is applied to the attitude/momentum control for the Space Station that uses control moment gyros (CMGs) as the primary actuating devices and gravity gradient torque to manage momentum stored in CMGs. The moments of inertia of the Space Station are assumed to be constant but uncertain. In Refs. 2 and 3 the linear quadratic regulator (LQR) design procedure has been used to control the attitude/momentum of the Space Station. Full state information is assumed since the attitude and attitude rate are assumed to be very accurately measured. In Ref. 2 disturbance rejection filters are augmented to the system to handle the external cyclic disturbance torque, and the LQR design and pole assignment procedures for pitch control and roll-yaw control, respectively, are applied to the augmented system. In this paper the system equation is differentiated until the external disturbance torque term disappears in the resulting equation to apply the design procedure developed in Sec. II. The resulting controller consists of conventional proportional-integral-derivative (PID) control and the cyclic disturbance rejection filter as in Ref. 2.

The application of the game theoretic approach combined with the internal feedback loop decomposition for describing parameter uncertainty allows very large variation in the inertia of the Space Station with little deterioration in performance. In Ref. 4, a differential game approach to developing synthesis techniques was taken where the parameter uncertainty was not decomposed and only the uncertainty in the system matrix is considered. In Refs. 5–7, Lyapunov stability theory has been used to design a control law for a system with uncertainty. This approach is similar to that used here in that a particular algebraic Riccati equation (ARE) must be solved. In Ref. 8, by adopting an input-output decomposition of the parameter uncertainty, the uncertain system is represented as an internal feedback loop (IFL) in which the parameter uncertainty is embedded in the system as a fictitious disturbance. Tahk and Speyer[8] developed the parameter robust linear-quadratic Gaussian (PRLQG) synthesis procedure that is an LQG design based on an extension of loop transfer recovery for the IFL description. In Refs. 7 and 8, the system is augmented to accommodate the input matrix uncertainty. Theoretically, this approach is limited in that input and output matrices associated with the IFL are to have the dimension of the original input and outputs, respectively. In the game theoretic approach, this restriction is not required. By considering the input and fictitious input in the IFL description as two noncooperative players, a finite-time linear differential game problem is constructed. By taking the quadratic norm of the fictitious output, the cost criterion is augmented by a term that emphasized robust performance. By taking the limit to an infinite-time, time-invariant linear system, a time-invariant control law is obtained. It is shown that the resulting time-invariant controller stabilizes the uncertain system for a prescribed parameter uncertainty bound. The development of the game theoretic controller is presented in Sec. II. The approach taken in Ref. 1 generalizes the results here to the partial information problem where only some noisy measurements of the states are available.

One motivation for this paper is to demonstrate on a meaningful problem the design process using the game theoretic controller augmented with the IFL decomposition. Although the linear quadratic regulator has guaranteed gain and phase margin, many systems remain sensitive to parameter variations. This control problem is particularly interesting in that the variation in the moments of inertia are bounded by physical constraint. The IFL decomposition allows selective changes in the moments of inertia to be included in the design process. In the pitch channel, there are two independent parameter uncertainties, one associated with the system matrix and the other associated with the input matrix. The quadratic norm of the fictitious output from the IFL decomposition of the system matrix augments the quadratic cost criterion and this augmented term represents a measure of system robustness. The effect of the fictitious inputs by the decomposition of the input matrix is to increase the gain. However, in the roll-yaw axis where there are three independent parameters, stability robustness in directions associated with inertia variations that can be made large before reaching physical constraint is achieved without increased bandwidth. The essential design task is choosing the weighting for combining the parameter uncer-

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tainty directions that improve stability robustness subject to the physical constraints on the inertias.

This work is based upon reports in Refs. 9 and 10 where the internal feedback loop concept[8] was applied to the problem of improving robustness of momentum management and attitude control in the roll-yaw axes using LQR theory. The game theoretic controller[1] first suggested in Ref. 11 was first applied to this problem in Ref. 12 for the pitch axis. At that time the authors became aware of the work in Ref. 13 for the roll-yaw axes using similar techniques, and later for all axes in Refs. 14 and 15.

II. Game-Theoretic Controller

A controller for a linear time-invariant system with parameter uncertainties in the system and input matrices is derived via the differential game frame work. A game theoretic approach is taken because it is shown under certain conditions that there exists a non-negative definite solution to an ARE, then the disturbance attenuation function is bounded.[1,22] This is equivalent to imposing an H_∞ norm bound on the transfer function between the disturbance input and the desired output.[1,22] In this section the disturbance inputs associated with system parameter uncertainty are constructed by the internal feedback loop decomposition of Refs. 7 and 8.

Consider a time-invariant linear system with uncertainties in the system and input matrices described by

$$\dot{x} = (A_0 + \Delta A)x + (B_0 + \Delta B)u \tag{1}$$

where $x \in R^n$, $u \in R^m$, $A_0 \in R^{n\times n}$, and $B_0 \in R^{n\times m}$ denote the state, the input, the nominal system matrix, and the nominal input matrix, respectively, and ΔA and ΔB are perturbations of the system matrix and the input matrix, respectively, due to parameter variations. It is assumed that all states are directly measured and (A_0, B_0) is a stabilizable pair.

By adopting the input-output decomposition modeling[8] of the perturbations, ΔA and ΔB are represented as

$$\Delta A = DL_a(\epsilon)E, \qquad \Delta B = FL_b(\epsilon)G \tag{2}$$

where ϵ denotes the parameter variation vector that is constant but unknown, and D, E, F, and G are known constant matrices. It is noted that the elements of ϵ need not be independent of each other. With this modeling of ΔA and ΔB, the uncertain dynamic system in Eq. (1) can be represented as an internal-feedback-loop description[8] in which the system is assumed to be forced by fictitious disturbances caused by the parameter uncertainty:

$$\dot{x} = A_0 x + B_0 u + \Gamma w \tag{3}$$

$$y_1 = \begin{bmatrix} E \\ 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ G \end{bmatrix} u \tag{4}$$

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} L_a(\epsilon) & 0 \\ 0 & L_b(\epsilon) \end{bmatrix} y_1 \tag{5}$$

where $w = [w_1^T \;\; w_2^T]^T$ is the fictitious disturbance and $\Gamma = [D \;\; F]$.

In the previous IFL description, the fictitious disturbance w is a feedback signal of y_1 amplified by the unknown gain

$$\begin{bmatrix} L_a(\epsilon) & 0 \\ 0 & L_b(\epsilon) \end{bmatrix}$$

Hence, one way to reduce the effect of parameter uncertainty is to assume that w is an independent Gaussian white noise and design a controller minimizing the cost

$$\lim_{t_f\to\infty} \frac{1}{t_f} \int_0^{t_f} (\rho y^T y + y_1^T y_1)\, dt$$

subject to the system in Eq. (3) where y is a performance measure defined as

$$y = \begin{bmatrix} C \\ 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ C_1 \end{bmatrix} u \tag{6}$$

and ρ is a positive constant that represents the tradeoff between the performance described by $y^T y$ and the robustness with respect to parameter uncertainty described by $y_1^T y_1$. Let

$$Q = \rho C^T C + E^T E$$

$$R = \rho C_1^T C_1 + G^T G$$

By assuming (Q, A_0) is detectable, R is positive definite, and $G = 0$, this cost criterion leads to the PRLQG design procedure[8] as $\rho \to 0$.

An alternate approach to robust synthesis is to design a controller to make the disturbance attenuation function due to the fictitious disturbance bounded, i.e.,

$$\sup_{w\in L_2[0,t_f]} \frac{\int_0^{t_f} (\rho Y^T y + y_1^T y_1)\, dt}{\int_0^{t_f} w^T w\, dt} < \gamma^{-2}$$

where γ is a positive constant, and t_f is a fixed final time. This problem can be solved by solving a differential game[16,22] to find u that minimizes and w that maximizes the cost criterion

$$J(u,w,t_f) = \int_0^{t_f} (y^T y + y_1^T y_1 - \gamma^{-2} w^T w)\, dt \tag{7}$$

subject to Eq. (3). It is well known[4,17] that if there exists a real symmetric solution $\Pi(t)$ over the interval $t \in [0, t_f]$ to the Riccati differential equation (RDE)

$$-\dot{\Pi} = A_0^T \Pi + \Pi A_0 - \Pi(B_0 R^{-1} B_0^T - \gamma^2 \Gamma\Gamma^T)\Pi + Q$$

with the final condition $\Pi(t_f) = 0$, then the strategies for u and w described as

$$u^* = -R^{-1} B^T \Pi(t) x$$

$$w^* = \gamma^{-2} \Gamma^T \Pi(t) x$$

yield the saddle point, i.e.,

$$J(u^*, w, t_f) \le J(u^*, w^*, t_f) \le J(u, w^*, t_f) \tag{8}$$

$\forall u, w \in L_2[0, t_f]$.

For the case where $t_f \to \infty$, $\Pi(t)$ converges to a constant matrix if there exists a non-negative definite solution to the algebraic Riccati equation

$$0 = A_0^T \bar{\Pi} + \bar{\Pi} A_0 - \bar{\Pi}(B_0 R^{-1} B_0^T - \gamma^2 \Gamma\Gamma^T)\bar{\Pi} + Q \tag{9}$$

Note that in general there may be many non-negative definite solutions to the ARE, Eq. (9). The minimal non-negative definite solution[1,6] to the ARE, Eq. (9), denoted as $\bar{\Pi}$, is defined as a non-negative definite solution to the ARE, Eq. (9), such that $\bar{\Pi} \le \bar{\Pi}$ where $\bar{\Pi}$ is any non-negative definite solution to the ARE, Eq. (9). Then $\Pi(t) \to \bar{\Pi}$ as $t_f \to \infty$.[1,17] Hence, u^* and w^* become time-invariant strategies described by

$$\bar{u} = -R^{-1} B^T \bar{\Pi} x \tag{10a}$$

$$\bar{w} = \gamma^{-2} \Gamma^T \bar{\Pi} x \tag{10b}$$

The resulting time-invariant strategies in Eq. (10), however, may not satisfy the right-hand inequalities in Eq. (8) as $t_f \to \infty$.[17] However, only the left-hand inequality is of concern in the development of this class of robust controllers.

In the worst case design, since the fictitious disturbance w is not an intelligent player, only the control strategy for the control u given by Eq. (10a) can be implemented. The following proposition provides a robustness property for the control law in Eq. (10a).

A. Proposition 1

Assume that R is a positive definite matrix and (Q, A_0) is a detectable pair. Suppose that there exists a non-negative definite solution $\bar{\Pi}$ to the ARE, Eq. (9). Then, the control law given as

$$u = -R^{-1}B^T\bar{\Pi}x \tag{11}$$

stabilizes the uncertain dynamic system in Eq. (1) for all ϵ such that $\|L_a(\epsilon)\| < \gamma$, and $\|L_b(\epsilon)\| < \gamma$.

B. Claim 1

Suppose that $D_1^TD_1 + G^TG > 0$. Then,

$$D_1^TU_1D_1 + G^TU_2G > 0 \qquad \forall U_1, U_2 > 0$$

Proof: It is sufficient to prove $D_1^TU_1D_1 + G^TU_2G$ is nonsingular. Suppose that there exists a nonzero vector z such that

$$z^T(D_1^TU_1D_1 + G^TU_2G)z = 0$$

Then, $D_1z = 0$ and $Gz = 0$ since U_1 and U_2 are positive definite, hence $(D_1^TD_1 + G^TG)z = 0$ that contradicts the assumption. □

Proof of Proposition 1: By using the control law in Eq. (11), the closed-loop system is described as

$$\dot{x} = A_cx \tag{12}$$

where

$$A_c = A_0 + DL_a(\epsilon)E - \left[B_0 + FL_b(\epsilon)G\right]R^{-1}B_0^T\bar{\Pi}$$

The ARE, Eq. (9), can be rewritten as the following Lyapunov equation:

$$A_c^T\bar{\Pi} + \bar{\Pi}A_c = -Q_1 \tag{13}$$

where

$$\begin{aligned} Q_1 &= \bar{\Pi}B_0R^{-1}\Delta_bR^{-1}B_0^T\bar{\Pi} + E^T\Delta_aE + \rho C^TC \\ &\quad + \gamma^2(\bar{\Pi}D - \gamma^{-2}E^TL_a^T)(\bar{\Pi}D - \gamma^{-2}E^TL_a^T)^T \\ &\quad + \gamma^{-2}\bar{\Pi}(\gamma^2F + B_0R^{-1}G^TL_b^T)(\gamma^2F + B_0R^{-1}G^TL_b^T)^T\bar{\Pi} \end{aligned}$$

$$\Delta_a = I - \gamma^{-2}L_a(\epsilon)^TL_a(\epsilon)$$

$$\Delta_b = \rho C_1^TC_1 + G^T[I - \gamma^{-2}L_b(\epsilon)^TL_b(\epsilon)]G$$

and where $\|L_a(\epsilon)\| < \gamma$ implies that $\Delta_a > 0$, and $\|L_b(\epsilon)\| < \gamma$ and claim 1 yield $\Delta_b > 0$. Hence, Q_1 is non-negative definite. Now it will be shown that (Q_1, A_c) is a detectable pair by contradiction. Suppose (Q_1, A_c) is not detectable. Then there exists a nonzero vector z for some s in the closed right half plane such that $(sI - A_c)z = 0$ and $Q_1z = 0$. Since each term in Q_1 is non-negative definite, $z^TQ_1z = 0$ leads to

$$z^T(\bar{\Pi}B_0R^{-1}\Delta_bR^{-1}B_0^T\bar{\Pi} + E^T\Delta_aE + \rho C^TC)z = 0$$

which implies that $B_0^T\bar{\Pi}z = 0$, $Ez = 0$, and $Cz = 0$, hence

$$(sI - A_c)z = (sI - A_0)z$$

Therefore,

$$\begin{bmatrix} sI - A_0 \\ \rho C^TC + E^TE \end{bmatrix} z = 0$$

which contradicts the assumption (Q, A_0) is detectable. Applying the lemma 4.2[18] to the Lyapunov equation, Eq. (13), completes the proof.

Note that proposition 1 holds for any non-negative solution to the ARE, Eq. (9). However, the minimal non-negative solution $\bar{\Pi}$ produces the smaller gain for the control law. To design the controller, Eq. (11), the design parameters ρ and γ should be chosen for the ARE, Eq. (9), to have a non-negative definite solution. In particular, as the value of ρ increases, system performance improves but the stability robustness with respect to the parameter variation becomes poor. As the value of γ increases, stability robustness with respect to parameter variation improves.

III. Space Station Control

The game theoretic controller developed in Sec. II is applied to the attitude/momentum control for the Space Station.

A. Space Station Dynamics

The Space Station is expected to maintain a local vertical/local horizontal (LVLH) orientation during normal operation. Suppose that the Space Station control (body) axes are aligned with the principal axes. (For the phase I configuration of the Space Station, this is a good assumption.[2]) For the small deviation from the LVLH frame, the linearized Space Station dynamics are described as[2,3]

$$\ddot{\phi} + 4\omega_0^2k_x\phi - \omega_0(1 - k_x)\dot{\psi} = -\frac{1}{I_x}(T_x - w_x) \tag{14a}$$

$$\ddot{\theta} - 3\omega_0^2k_y\theta = -\frac{1}{I_y}(T_y - w_y) \tag{14b}$$

$$\ddot{\psi} - \omega_0^2k_z\psi + \omega_0(1 + k_z)\dot{\phi} = -\frac{1}{I_z}(T_z - w_z) \tag{14c}$$

where the body fixed axes (x, y, z) denote the roll, pitch, and yaw control axes with the roll axis in flight direction, the pitch axis normal to the orbit plane, and the yaw axis toward the Earth; ϕ, θ, and ψ denote the roll, pitch, and yaw Euler angles with respect to the LVLH frame; (T_x, T_y, T_z) is the control torque vector produced by the CMG with respect to the control axes; (w_x, w_y, w_z) is an external disturbance torque vector with respect to the control axes; ω_0 is the orbital rate of 0.0011 rad/s; and k_x, k_y, and k_z are the parameters defined from the moments of inertia I_x, I_y, and I_z as

$$k_x = \frac{I_y - I_z}{I_x}, \qquad k_y = \frac{I_z - I_x}{I_y}, \qquad k_z = \frac{I_x - I_y}{I_z} \tag{15}$$

Terms involving ω_0^2 in Eq. (14) represent the combined gravity gradient and gyroscopic torque in each axis. The CMG momentum dynamics are[2,3]

$$\dot{h}_x - \omega_0h_z = T_x \tag{16a}$$

$$\dot{h}_y = T_y \tag{16b}$$

$$\dot{h}_z + \omega_0h_x = T_z \tag{16c}$$

where (h_x, h_y, h_z) is the CMG momentum vector with respect to the control axes. It is assumed that the Euler angle, the Euler angle rate, and the CMG momentum are perfectly measured. The roll and yaw dynamics are coupled while the pitch axis is uncoupled.

The physical constraints for the parameters k_x, k_y, and k_z due to the triangular inequality of moment of inertia[19] are

$$|k_i| < 1, \qquad i = x, y, z \tag{17}$$

Table 1 Variation type and physical limit of variation due to the triangular inequality

Variation type ($\Delta_i = [\Delta I_{xn}\ \Delta I_y\ \Delta I_z]$)	Physical bound of δ, %
$\Delta_1 = \delta[I_{xn}\ I_{yn}\ I_{zn}]$	$-100.0 < \delta < \infty$
$\Delta_2 = \delta[-I_{xn}\ I_{yn}\ I_{zn}]$	$-15.9 \le \delta \le 2.6$
$\Delta_3 = \delta[I_{xn}\ -I_{yn}\ I_{zn}]$	$-81.9 \le \delta \le 13.1$
$\Delta_4 = \delta[I_{xn}\ I_{yn}\ -I_{zn}]$	$-2.1 \le \delta \le 19.5$
$\Delta_5 = \delta[0\ I_{yn}\ I_{zn}]$	$-27.5 \le \delta \le 5.3$
$\Delta_6 = \delta[I_{xn}\ 0\ I_{zn}]$	$-90.0 \le \delta \le 30.3$
$\Delta_7 = \delta[I_{xn}\ I_{yn}\ 0]$	$-4.1 \le \delta \le 48.4$
$\Delta_8 = \delta[0\ -I_{yn}\ I_{zn}]$	$-40.0 \le \delta \le 3.6$
$\Delta_9 = \delta[-I_{xn}\ 0\ I_{zn}]$	$-17.5 \le \delta \le 2.3$
$\Delta_{10} = \delta[-I_{xn}\ I_{yn}\ 0]$	$-31.3 \le \delta \le 6.4$

The moments of inertia I_x, I_y, and I_z are assumed to be constant but uncertain and described by

$$I_i = I_{in} + \Delta I_i, \qquad i = x,y,z \tag{18}$$

where the subscript n and ΔI_i denote the nominal value and the variation of each moment of inertia, respectively. Then the parameters k_x, k_y, and k_z can be represented as

$$k_i = k_{in} + \Delta k_i, \qquad i = x,y,z \tag{19}$$

where k_{in} denotes the value of k_i with nominal values of the moments of inertia and Δk_i denotes the variation due to the variation of the moments of inertia. The nominal values of the moments of inertia for the phase I configuration are

$$I_{xn} = 50.28\text{E}6, \qquad I_{yn} = 10.80\text{E}6, \qquad I_{zn} = 58.57\text{E}6$$

in unit of slug-ft^2. In order to check the stability margin for inertia variation, 10 types of variations of the moments of inertia listed in Table 1 are considered. Each variation is limited by the physical constraint in Eq. (17). Table 1 also shows the physical limit of each type of variation.

The external disturbances (w_x, w_y, w_z) are modeled as[2,3]

$$w_i = A_{1i}^d \sin(\omega_o t + \varphi_{1i}) + A_{2i}^d \sin(2\omega_o t + \varphi_{2i}) + B_i^d \tag{20}$$

where $i = x,y,z$ when A_{1i}^d, A_{2i}^d, and B_i^d are assumed to be constant but unknown. The cyclic aerodynamic disturbance at orbital rate and twice the orbital rate are due to the diurnal bulge and the rotating solar panel, respectively.

B. Pitch Control

Before developing the controller for the pitch axis, the open-loop characteristics of the pitch channel are investigated. For the external disturbance free case, suppose the constant feedback control described as

$$T_y = [K_1 \quad K_2 \quad K_3]\begin{bmatrix}\theta \\ \dot{\theta} \\ h_y\end{bmatrix}$$

stabilizes the nominal system of Eqs. (14b) and (16b). By using this control, the closed-loop characteristic equation $\Delta(s)$ becomes

$$\Delta(s) = s^3 - \left(K_3 - \frac{K_2}{I_y}\right)s^2 + \left(\frac{K_1}{I_y} - 3\omega_0^2 k_y\right)s + 3\omega_0^2 K_3 k_y$$

By the Routh-Hurwitz criterion,[21] the zeroth-order term on the right side should be positive for $k_y = k_{yn}$. Hence, the given feedback control law cannot stabilize the system when the sign of k_y is different from that of k_{yn}. In other words, any constant feedback control law designed for the nominal system cannot stabilize the system with a k_y whose nominal value has a different sign.

The appearance of a cyclic disturbance, described in Ref. 20, prevents the direct application of the game theoretic controller to the pitch channel. However, this can be avoided by differentiating Eqs. (14b) and (16b) until the cyclic disturbance term disappears in the resulting equation. Differentiating Eqs. (14b) and (16b) five times yields

$$\theta^{(\text{VII})} = (3k_y - 5)\omega_0^2\theta^{(\text{V})} + (15k_y - 4)\omega_0^4\theta^{(\text{III})} + 12k_y\omega_0^6\dot{\theta} - f_y u_y \tag{21}$$

$$h_y^{(\text{VI})} = -5\omega_0^2 h_y^{(\text{IV})} - 4\omega_0^4\ddot{h}_y + I_{yn}u_y \tag{22}$$

where the parenthetical superscripts represent the order of the time derivative, $f_y = (I_{yn}/I_y)$, and u_y is a new control variable defined as

$$u_y = \frac{1}{I_{yn}}\left(T_y^{(\text{V})} + 5\omega_0^2 T_y^{(\text{III})} + 4\omega_0^4\dot{T}_y\right) \tag{23}$$

Note that the parameter f_y has uncertainty due to the uncertainty in I_y and is represented as

$$f_y = 1 + \Delta f_y \tag{24}$$

Equations (21) and (22), however, are not yet an adequate representation of the system equation for the design procedure developed in Sec. II since they contain uncontrollable modes on the imaginary axis when u_y is used as a control. It can be verified that the uncontrollable modes are at $s = 0$, $s = \pm j\omega_0$, and $s = \pm 2j\omega_0$, which arise from differentiation of the cyclic disturbance. The uncontrollability problem can be avoided by changing the regulated variables. The uncontrollable mode at $s = 0$ can be removed by regulating $\dot{\theta}$ instead of θ. If $\dot{h}_y$ is regulated instead of h_y, h_y becomes unbounded as time increases. It is clear from original pitch dynamics and CMG momentum equations that the pitch attitude θ cannot be regulated since T_y in Eq. (14b) requires a biased control to regulate θ in steady state under the disturbance w_y. Hence, h_y becomes unbounded. Note that regulating $\dot{h}_y$ instead of h_y still produces the uncontrollable mode at $s = 0$. Since the uncontrollable oscillating modes at $s = \pm j\omega_0$ and $s = \pm 2j\omega_0$ arise in Eqs. (21) and (22), all the modes of the θ aud h_y channels cannot be regulated. However, as will be shown, the oscillating modes in either the θ or h_y channel can be regulated. To regulate $\dot{\theta}$ instead of h_y, a new state ξ_y, defined as

$$\xi_y = h_y^{(\text{IV})} + 5\omega_0^2\ddot{h}_y + 4\omega_0^4 h_y \tag{25}$$

is regulated. Thereby, the uncontrollable mode at $s = \pm j\omega_0$ and $s = \pm 2j\omega_0$ are embedded in ξ_y. In a similar way, for regulating h_y, a new state ξ_θ, defined as

$$\xi_\theta = \theta^{(\text{V})} + 5\omega_0^2\theta^{(\text{III})} + 4\omega_0^4\dot{\theta} \tag{26}$$

is regulated. The result is that h_y or θ becomes a harmonic function with angular rates ω_0 and $2\omega_0$ in steady state. In this paper, only the design for regulating $\dot{\theta}$ is considered, since the development of the control law for regulating h_y is similar.

From Eq. (22) ξ_y satisfies

$$\ddot{\xi}_y = I_{yn}u_y \tag{27}$$

By defining a state vector x_y as

$$x_y \triangleq \left[\dot{\theta} \quad \ddot{\theta} \quad \theta^{(\text{III})} \quad \theta^{(\text{IV})} \quad \theta^{(\text{V})} \quad \theta^{(\text{VI})} \quad \xi_y \quad \dot{\xi}_y\right]^T$$

Eqs. (21) and (27) can be represented as

$$\dot{x}_y = (A_{0y} + \Delta A_y)x_y + (B_{0y} + \Delta B_y)u_y \tag{28}$$

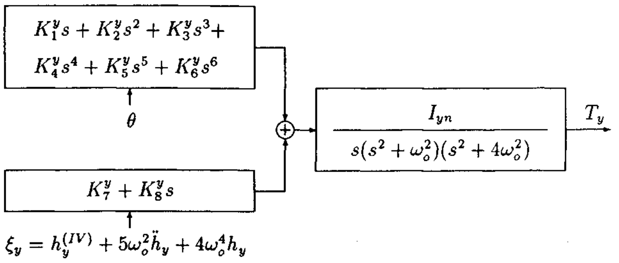


Fig. 1 Block diagram of control law for the pitch axis.

where

$$A_{0y} = \left[\begin{array}{cccccc|cc} 0 & 1 & 0 & 0 & 0 & 0 & & \\ 0 & 0 & 1 & 0 & 0 & 0 & & \\ 0 & 0 & 0 & 1 & 0 & 0 & \multicolumn{2}{c}{0_{6\times 2}} \\ 0 & 0 & 0 & 0 & 1 & 0 & & \\ 0 & 0 & 0 & 0 & 0 & 1 & & \\ 12k_{yn}\omega_0^6 & 0 & k_{y1}\omega_0^4 & 0 & k_{y2}\omega_0^2 & 0 & & \\ \multicolumn{6}{c}{0_{2\times 6}} & 0 & 1 \\ & & & & & & 0 & 0 \end{array}\right]$$

$$\Delta A_y = \begin{bmatrix} & & & 0_{5\times 8} & & & & \\ 12\Delta k_y\omega_0^6 & 0 & 15\Delta k_y\omega_0^4 & 0 & 3\Delta k_y\omega_0^2 & 0 & 0 & 0 \\ & & & 0_{2\times 8} & & & & \end{bmatrix}$$

$$B_{0y} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -1 & 0 & I_{yn} \end{bmatrix}^T$$

$$\Delta B_y = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -\Delta f_y & 0 & 0 \end{bmatrix}^T$$

$$k_{y1} = 15k_{yn} - 4, \qquad k_{y2} = 3k_{yn} - 5$$

Note that the pitch angle θ is not included in the state vector x_y. The variations ΔA_y and ΔB_y can be decomposed as

$$\Delta A_y = D_y L_{ay}(n\Delta k_y)E_y$$

$$\Delta B_y = F_y L_{by}(\Delta f_y)G_y$$

where

$$D_y = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}^T$$

$$E_y = \begin{bmatrix} 12\dfrac{\omega_0^6}{n} & 0 & 15\dfrac{\omega_0^4}{n} & 0 & 3\dfrac{\omega_0^2}{n} & 0 & 0 & 0 \end{bmatrix}$$

$$F_y = -D_y, \qquad G_y = [1]$$

$$L_{ay} = [n\Delta k_y], \qquad L_{by} = [\Delta f_y]$$

The parameter n in the previous equation denotes the weighting between Δk_y and Δf_y.

The control law can be obtained by identifying $[n\Delta k_y, \Delta f_y]^T$ as ϵ in Eq. (2), dropping the subscript y, and using Eqs. (9) and (11) with appropriate choices of ρ, γ, n, C, and C_1. Then the control law u_y is represented in the form of

$$u_y = K_y x_y \tag{29}$$

where K_y is control gain matrix. From the definitions of u_y and ξ_y, Eq. (29) becomes

$$\begin{aligned} \frac{1}{I_{yn}}\left(T_y^{(V)} + 5\omega_0^2 T_y^{(III)} + 4\omega_0^4 \dot{T}_y\right) &= K_1^y\dot{\theta} + K_2^y\ddot{\theta} \\ &+ K_3^y\theta^{(III)} + K_4^y\theta^{(IV)} + K_5^y\theta^{(V)} + K_6^y\theta^{(VI)} \\ &+ K_7^y\left(h_y^{(IV)} + 5\omega_0^2\ddot{h}_y + 4\omega_0^4 h_y\right) \\ &+ K_8^y\left(h_y^{(V)} + 5\omega_0^2 h_y^{(III)} + 4\omega_0^4\dot{h}_y\right) \end{aligned} \tag{30}$$

where K_i^y denotes the ith element of the gain matrix K_y. The previous form is not realizable since it needs derivatives of $\dot{\theta}$ and h_y. Figure 1 describes Eq. (30). Define a new variable χ as

$$\chi_y = \frac{1}{I_{yn}} T_y - K_5^y\theta - K_6^y\dot{\theta} - K_7^y\int h_y \,\mathrm{d}t - K_8^y h_y \tag{31}$$

Then Eq. (30) becomes

$$\begin{aligned} &\chi_y^{(V)} + 5\omega_0^2\chi_y^{(III)} + 4\omega_0^4\dot{\chi}_y \\ &\quad = (K_4^y - 5\omega_0^2 K_6^y)\theta^{(IV)} + (K_3^y - 5\omega_0^2 K_5^y)\theta^{(III)} \\ &\quad + (K_2^y - 4\omega_0^4 K_6^y)\ddot{\theta} + (K_2^y - 4\omega_0^4 K_6^y)\dot{\theta} \end{aligned} \tag{32}$$

The previous equation can be implemented by using the canonical realizations such as the controller canonical realization, the observer canonical realization, and the parallel canonical realization.[20] In this paper, the parallel canonical realization is adopted. Introduce variables ζ_y and η_y such that

$$\ddot{\zeta}_y + \omega_0^2\zeta_y = \theta \tag{33a}$$

$$\ddot{\eta}_y + 4\omega_0^2\eta_y = \theta \tag{33b}$$

Then χ_y can be represented in terms of ζ and η as

$$\chi_y = \mathcal{A}_y\zeta_y + \mathcal{B}_y\dot{\zeta}_y + \mathcal{C}_y\eta_y + \mathcal{D}_y\dot{\eta}_y \tag{34}$$

where

$$\mathcal{A}_y = \tfrac{1}{3}(\omega_0^2 K_5^y - K_3^y + \omega_0^{-2}K_2^y)$$

$$\mathcal{B}_y = \tfrac{1}{3}(\omega_0^2 K_6^y - K_4^y + \omega_0^{-2}K_2^y)$$

$$\mathcal{C}_y = -\tfrac{1}{3}(16\omega_0^2 K_5^y - 4K_3^y + \omega_0^{-2}K_1^y)$$

$$\mathcal{D}_y = -\tfrac{1}{3}(16\omega_0^2 K_6^y - 4K_4^y + \omega_0^{-2}K_2^y)$$

Combining Eqs. (31) and (34) yields an implementable form for the control torque T_y as

$$\begin{aligned} T_y &= \bar{K}_1^y\theta + \bar{K}_2^y\dot{\theta} + \bar{K}_3^y\int h_y \,\mathrm{d}t + \bar{K}_4^y h_y \\ &+ \bar{K}_5^y\zeta_y + \bar{K}_6^y\dot{\zeta}_y + \bar{K}_7^y\eta_y + \bar{K}_8^y\dot{\eta}_y \end{aligned} \tag{35}$$

where

$$\bar{K}_1^y = I_{yn}K_5^y, \quad \bar{K}_2^y = I_{yn}K_6^y, \quad \bar{K}_3^y = I_{yn}K_7^y, \quad \bar{K}_4^y = I_{yn}K_8^y$$

$$\bar{K}_5^y = I_{yn}\mathcal{A}_y, \quad \bar{K}_6^y = I_{yn}\mathcal{B}_y, \quad \bar{K}_7^y = I_{yn}\mathcal{C}_y, \quad \bar{K}_8^y = I_{yn}\mathcal{D}_y$$

The control torque described by Eq. (35) forms a dynamical feedback control law that has the same form as in Ref. 2. Figure 2 describes the control law in Eq. (35). The integral feedback in Eq. (35) is expected to reject the constant input disturbance as in classical control theory.[21] Equation (33) represents the cyclic disturbance rejection filter for attitude hold in the pitch axis. The initial states of the integrator in Eq. (35)

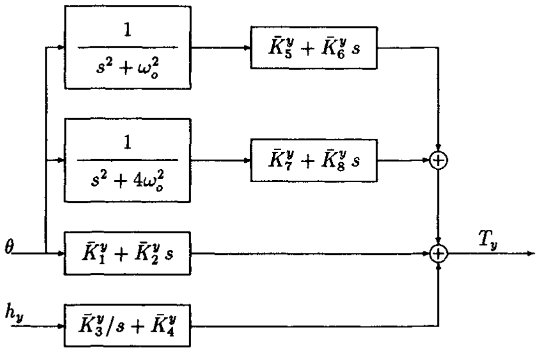
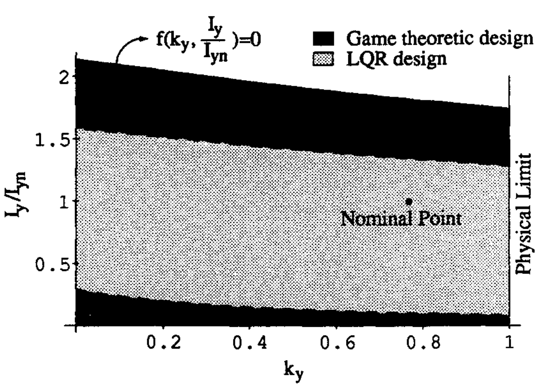


Fig. 2 Realizable form of control law for the pitch axis.



$$f(k_y, \frac{I_y}{I_{yn}}) = \left(\frac{I_y}{I_{yn}}\right)^{-3} - \left(\frac{I_y}{I_{yn}}\right)^{-2}(0.6722\,k_y + 0.8039)$$
$$+ \left(\frac{I_y}{I_{yn}}\right)(0.1400\,k_y^2 + 0.3980\,k_y + 0.1800)$$
$$- 0.0085\,k_y^3 - 0.0476\,k_y^2 - 0.0482\,k_y - 0.0105$$

Fig. 3 Comparison of stable region for the pitch control.

and the cyclic disturbance rejection filter are the designer's choice.

For the controller design, ρ, γ, C_1, and C are chosen as

$$\rho = 0.81, \qquad \gamma = 0.2, \qquad n = 5, \qquad C_1 = 0$$

$$C = \text{diag}\left(3.9\omega_0^6 \;\; 3.9\omega_0^5 \;\; 3.9\omega_0^4 \;\; 3.9\omega_0^3 \;\; 3.9\omega_0^2 \;\; 3.9\omega_0 \;\; \frac{\omega_0^2}{I_{yn}} \;\; \frac{\omega_0}{I_{yn}}\right)$$

and the minimal non-negative definite solution to the corresponding ARE is taken. Table 2 shows the controller gain matrix $\bar{K}^y$ and the closed-loop eigenvalues. A stable region for the system parameters of the game theoretic design shown in Fig. 3 is obtained by applying the Routh-Hurwitz criterion to the closed-loop system for the given control law. MATHEMATICA™ software is used to check the Routh-Hurwitz criterion. Stability margins in some specific direction are listed in Table 3. The stable region of the game theoretic design and the LQR design in Ref. 2 are compared in Fig. 3. The bound $k_y = 0$ comes from the open-loop characteristic. Figure 3 shows that the game theoretic design improves the stability robustness with respect to the parameter variations. A simulation is performed with parameter set considered in Ref. 2

$$A_{1y}^d = 2 \text{ ft-lb}, \qquad A_{2y}^d = 0.5 \text{ ft-lb}, \qquad B_y^d = 4 \text{ ft-lb}$$

$$\varphi_{1y} = 0 \text{ deg}, \qquad \phi(0) = 1 \text{ deg}, \qquad \dot{\phi}(0) = 0.001 \text{ deg/s}$$

and other initial conditions = 0. Figure 4 shows the time responses for the nominal system and a perturbed system with

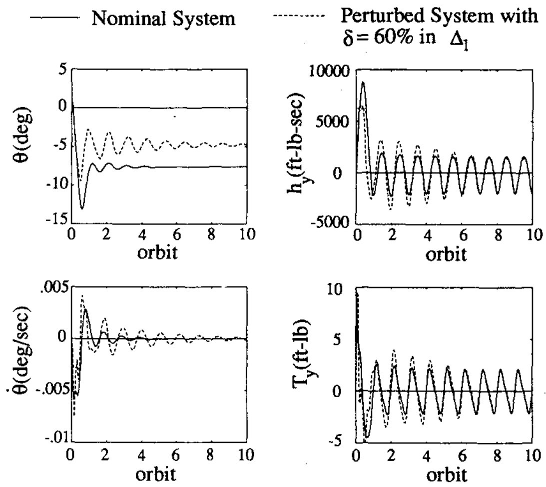


Fig. 4 Time response for the pitch axis.

Table 2 Controller gains and closed-loop eigenvalues for the pitch channel

i	$\bar{K}_i^y$	Unit
1	3.9142E+2	ft-lb/rad
2	1.7736E+5	ft-lb-s/rad
3	1.9885E−6	ft-lb/ft-lb-s^2
4	6.6961E−3	ft-lb/ft-lb-s
5	2.7357E−6	ft-lb-rad/s^2
6	4.9017E−2	ft-lb-rad/s
7	−2.2665E−4	ft-lb-rad/s^2
8	2.0948E−1	ft-lb-rad/s
Closed-loop eigenvalues		
−4.77, −1.52,		−0.55±0.42j
−0.13±1.00j		−0.59±1.99j

$\delta = 60\%$ in Δ_1 variation denoted by solid line and dotted line, respectively. As expected, the attitude approaches torque equilibrium attitude (TEA), −7.6 deg for nominal system, while the CMG momentum oscillates with zero mean value in steady state.

C. Roll-Yaw Control

The controller for the roll-yaw axes is developed in a way similar to that for the pitch axis.

Define

$$e_x = \phi - \phi_c, \qquad e_z = \psi - \psi_c$$

where ϕ_c and ψ_c are the command roll and yaw attitude, respectively, and are assumed to be constant. Representing Eqs. (14a) and (14c) in terms of e_x and e_z and differentiating the resulting equations along with Eqs. (16a) and (16c) yield

$$e_x^{(\text{VII})} = -(5+4k_x)\omega_0^2 e_x^{(\text{V})} - 4(1+5k_x)\omega_0^4 e_x^{(\text{III})} - 16k_x\omega_0^6\dot{e}_x$$
$$+ (1-k_x)\omega_0(e_z^{(\text{VI})} + 5\omega_0^2 e_z^{(\text{IV})} + 4\omega_0^4\ddot{e}_z) - f_x u_x \tag{36a}$$

$$e_z^{(\text{VII})} = (k_z-5)\omega_0^2 e_z^{(\text{V})} + (5k_z-4)\omega_0^4 e_z^{(\text{III})} + 4k_z\omega_0^6\dot{e}_z$$
$$- (1+k_z)\omega_0(e_x^{(\text{VI})} + 5\omega_0^2 e_x^{(\text{IV})} + 4\omega_0^4\ddot{e}_x) - f_z u_z \tag{36b}$$

$$h_x^{(\text{VI})} = -5\omega_0^2 h_x^{(\text{IV})} - 4\omega_0^4\ddot{h}_x$$
$$+ \omega_0(h_z^{(\text{V})} + 5\omega_0^2 h_z^{(\text{III})} + 4\omega_0^5\dot{h}_z) + I_{xn}u_x \tag{36c}$$

$$h_z^{(\text{VI})} = -5\omega_0^2 h_z^{(\text{IV})} - 4\omega_0^4\ddot{h}_z$$
$$- \omega_0(h_x^{(\text{V})} + 5\omega_0^2 h_x^{(\text{III})} + 4\omega_0^5\dot{h}_x) + I_{zn}u_z \tag{36d}$$

Table 3 Stability margin of variation

Variation type	Lower and upper margin of δ	
	Pitch, %	Roll-yaw, %
Δ_1	$-99 \le \delta \le 82$	$-90 \le \delta \le 75$
Δ_2	$-7^{a} \le \delta \le 19$	$-90 \le \delta \le 99$
Δ_3	$-99 \le \delta \le 99$	$-62 \le \delta \le 56$
Δ_4	$-22 \le \delta \le 7^{a}$	$-73 \le \delta \le 63$
Δ_5	$-14 \le \delta \le 33$	$-99 \le \delta \le 99$
Δ_6	$-99 \le \delta \le 99$	$-73 \le \delta \le 66$
Δ_7	$-51 \le \delta \le 16^{a}$	$-99 \le \delta \le 99$
Δ_8	$-14^{a} \le \delta \le 43$	$-62 \le \delta \le 66$
Δ_9	$-7^{a} \le \delta \le 20$	$-74 \le \delta \le 89$
Δ_{10}	$-16^{a} \le \delta \le 37$	$-99 \le \delta \le 99$

[a] This bound comes from the open-loop characteristic in pitch axis.

Table 4 Rejection of the constant disturbance torque for the roll-yaw axes

Case	States	Uncontrollable mode in resulting system
1	e_x, e_z	none
2	h_x, h_z	none
3	e_x, h_x	none
4	e_z, h_z	none
5	e_x, h_z	$s=0$
6	h_x, e_z	$s=0$

Table 5 Rejection of the cyclic disturbance torque for the roll-yaw axes

Case	States	Uncontrollable mode in resulting system
1	e_x, e_z	$s = \pm j\omega_0$
2	h_x, h_z	none
3	e_x, h_x	$s = \pm j\omega_0$
4	e_z, h_z	$s = \pm 2j\omega_0$
5	e_x, h_z	$s = \pm j\omega_0$
6	h_x, e_z	none

where $f_x = (I_{xn}/I_x)$, $f_z = (I_{zn}/I_z)$, and u_x and u_z are new control variables defined as

$$u_i = \frac{1}{I_{in}}\left[T_i^{(V)} + 5\omega_0^2 T_i^{(III)} + 4\omega_0^5 \dot{T}_i\right], \qquad i = x,z$$

The system in Eq. (36) contains uncontrollable modes at $s=0$ (double pole), $s = \pm j\omega_0$ (double pole), and $s = \pm 2j\omega_0$ (double pole) that arise from the external disturbance torque. This means that the external constant disturbance torque and cyclic disturbance torque can be rejected in only two of the four states e_x, e_z, h_x, and h_z. In a way similar to that for the pitch control, these uncontrollable modes can be removed by changing the regulated variables. Tables 4 and 5 show the combination of two states in which the constant disturbance and the cyclic disturbance are rejected, respectively. As shown in Tables 4 and 5, an uncontrollable mode still exists in some of the outputs e_x, e_z, h_x, and h_z. Note that it is always e_x that does not reject the cyclic disturbance. However, contrary to the pitch channel where bias in pitch angle cannot be regulated, the bias in both yaw and roll angle can be rejected, leaving only an oscillation in the roll angle. In this paper, only case 1 for constant disturbance rejection and case 6 for cyclic disturbance rejection are considered.

To reject the constant disturbance torque in the attitude channels, the uncontrollable double poles at $s=0$ are embedded in the CMG-momentum channels by regulating $\dot{h}_x$ and $\dot{h}_z$ instead of h_x and h_z. Similarly, the uncontrollable double poles at $s = \pm j\omega_0$ and $s = \pm 2j\omega_0$ are embedded in the roll-attitude and yaw CMG-momentum channels to reject the cyclic disturbance torque in yaw-attitude and roll CMG-momentum channels. By defining ξ_ϕ and ξ_z as

$$\xi_\phi = e_x^{(IV)} + 5\omega_0^2 \ddot{e}_x + 4\omega_0^4 e_x$$

$$\xi_z = h_z^{(V)} + 5\omega_0^2 h_z^{(III)} + 4\omega_0^4 \dot{h}_z$$

Eq. (36) becomes

$$\xi_\phi^{(III)} = -4k_x\omega_0^2\dot{\xi}_\phi + (1-k_x)\omega_0 \times (e_z^{(VI)} + 5\omega_0^2 e_z^{(IV)} + 4\omega_0^4\ddot{e}_z) - f_x u_x \tag{37a}$$

$$e_z^{(VII)} = (k_z-5)\omega_0^2 e_z^{(V)} + (5k_z-4)\omega_0^4 e_z^{(III)} + 4k_z\omega_0^6\dot{e}_z - (1+k_z)\omega_0\ddot{\xi}_\phi - f_z u_z \tag{37b}$$

$$h_x^{(VI)} = -5\omega_0^2 h_x^{(IV)} - 4\omega_0^4\ddot{h}_x + \omega_0\xi_z + I_{xn}u_x \tag{37c}$$

$$\dot{\xi}_z = -\omega_0(h_x^{(V)} + 5\omega_0^2 h_x^{(III)} + 4\omega_0^5\dot{h}_x) + I_{zn}u_z \tag{37d}$$

By defining a state vector x as

$$x \triangleq \left[\xi_\phi\ \dot{\xi}_\phi\ \ddot{\xi}_\phi\ e_z\ \dot{e}_z\ \ddot{e}_z\ e_z^{(III)}\ e_z^{(IV)}\ e_z^{(V)}\ e_z^{(VI)}\ \dot{h}_x\ \ddot{h}_x\ h_x^{(III)}\ h_x^{(IV)}\ h_x^{(V)}\ \xi_z\right]^T$$

Eq. (37) can be rewritten as a state-space representation of the form

$$\dot{x} = Ax + B\begin{bmatrix} u_x \\ u_z \end{bmatrix} \tag{38}$$

where

$$A = \begin{bmatrix} A_a & 0_{10\times 6} \\ 0_{6\times 10} & A_m \end{bmatrix}$$

$$A_a = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & k_{x1} & 0 & 0 & 0 & k_{x2} & 0 & k_{x3} & 0 & k_{x4} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & k_{z1} & 0 & k_{z2} & 0 & k_{z3} & 0 & k_{z4} & 0 \end{bmatrix}$$

$$A_m = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -4\omega_0^4 & 0 & -5\omega_0^2 & 0 & \omega_0 \\ -4\omega_0^5 & 0 & -5\omega_0^3 & 0 & -\omega_0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0_{2\times 2} & \begin{matrix} -f_x \\ 0 \end{matrix} & 0_{2\times 6} & \begin{matrix} 0 \\ -f_z \end{matrix} & 0_{2\times 4} & \begin{matrix} 0 & I_{xn} \\ I_{zn} & 0 \end{matrix} \end{bmatrix}^T$$

$$k_{x1} = -4k_x\omega_0^2, \quad k_{x2} = 4(1-k_x)\omega_0^5, \quad k_{x3} = 5(1-k_x)\omega_0^3$$

$$k_{x4} = (1-k_x)\omega_0, \quad k_{z1} = -(1+k_z)\omega_0, \quad k_{z2} = 4k_z\omega_0^6$$

$$k_{z3} = (5k_z-1)\omega_0^4, \quad k_{z4} = (k_z-5)\omega_0^2$$

In the previous system, four system parameters, k_x, k_z, f_x, and f_z, are included. However, one of them is represented in terms of others; f_z is represented in terms of k_x, k_z, and f_x as

$$f_z = \frac{I_{zn}}{I_{xn}} \frac{1+k_z}{1-k_x} f_x$$

For small variations of k_x, k_y, and f_x, the variation of f_z, Δf_z is approximated as

$$\Delta f_z \approx \kappa_1 \Delta k_x + \kappa_2 \Delta k_z + \kappa_3 \Delta f_x$$

where

$$\kappa_1 = \frac{I_{zn}}{I_{xn}} \frac{(1+k_{zn})}{(1-k_{xn})^2}, \quad \kappa_2 = \frac{I_{zn}}{I_{xn}} \frac{1}{1-k_{xn}}, \quad \kappa_3 = \frac{I_{zn}}{I_{xn}} \frac{1+k_{zn}}{1-k_{xn}}$$

Then, the variation of the system and input matrices, ΔA and ΔB, can be decomposed as

$$\Delta A = DL_a(n_1 \Delta k_x, n_2 \Delta k_z)E$$

$$\Delta B = FL_b(n_1 \Delta k_x, n_2 \Delta k_z, \Delta f_x)G$$

where

$$L_a = \mathrm{diag}(n_1 \Delta k_x, n_2 \Delta k_z)$$

$$L_b = \mathrm{diag}(n_1 \Delta k_x, n_2 \Delta k_z, \Delta f_x)$$

$$D = \begin{bmatrix} 0_{2\times 2} & \begin{matrix} 1 \\ 0 \end{matrix} & 0_{2\times 6} & \begin{matrix} 0 \\ 1 \end{matrix} & 0_{2\times 6} \end{bmatrix}^T$$

$$E = \left(\frac{I_{yn}}{I_{xn}}\right)$$

$$\times \begin{bmatrix} 0 & 4\frac{\omega_0^2}{n_1} & 0 & 0 & 0 & 4\frac{\omega_0^5}{n_1} & 0 & 5\frac{\omega_0^3}{n_1} & 0 & \frac{\omega_0}{n_1} & \\ & & & & & & & & & & 0_{2\times 6} \\ 0 & 0 & -\frac{\omega_0}{n_2} & 0 & 4\frac{\omega_0^6}{n_2} & 0 & 5\frac{\omega_0^4}{n_2} & 0 & \frac{\omega_0^2}{n_2} & 0 & \end{bmatrix}$$

$$F = \begin{bmatrix} & 1 & & 0 & \\ & 0 & & 1 & \\ 0_{4\times 2} & 0 & 0_{4\times 6} & 1 & 0_{4\times 6} \\ & 0 & & 1 & \end{bmatrix}^T$$

$$G = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\kappa_1}{n_1} & \frac{\kappa_2}{n_2} & \kappa_3 \end{bmatrix}^T$$

The parameters n_1 and n_2 represent the weightings among the three system parameters. By the choice of n_1 and n_2 with the ratio of reciprocal of directional derivatives of k_x, k_z, and f_x, with respect to a particular inertia variation direction, the inertia variations listed in Table 1 can be assumed in the design process. The directions that are preferable for design are the inertia variations that can be made large before reaching a physical constraint.

The control law can be obtained by using Eqs. (9) and (11) with appropriate choices of ρ, γ, n_1, n_2, C, and C_1. Then, the control law u_y is represented in the form of

$$\begin{bmatrix} u_x \\ u_z \end{bmatrix} = Kx \tag{39}$$

where K is a control gain matrix. From the definitions of u_x, u_y, ξ_ϕ, and ξ_z, Eq. (39) becomes

$$\begin{aligned} \frac{1}{I_{in}} (T_i^{(\mathrm{V})} + 5\omega_0^2 T_i^{(\mathrm{III})} + 4\omega_0^4 \dot{T}_i) &= K_1^i \xi_\phi + K_2^i \dot{\xi}_\phi + K_3^i \ddot{\xi}_\phi \\ &+ K_4^i e_z + K_5^i \dot{e}_z + K_6^i \ddot{e}_z + K_7^i e_z^{(\mathrm{III})} + K_8^i e_z^{(\mathrm{IV})} + K_9^i e_z^{(\mathrm{V})} \\ &+ K_{10}^i e_z^{(\mathrm{VI})} + K_{11}^i \dot{h}_x + K_{12}^i \ddot{h}_x + K_{13}^i h_x^{(\mathrm{III})} \\ &+ K_{14}^i h_x^{(\mathrm{IV})} + K_{15}^i h_x^{(\mathrm{V})} + K_{16}^i \xi_z \end{aligned} \tag{40}$$

where $i = x,z$ when K_j^x and K_j^z denote the jth element of first row and second row of the gain matrix K, respectively. The previous form can be changed into a realizable form in a way similar to that done for the pitch control. Define a new variable χ_i as

$$\begin{aligned} \chi_i = \frac{1}{I_{in}} T_i - K_1^i \int e_x \, \mathrm{d}t - K_2^i e_x - K_3^i \dot{e}_x - K_9^i e_z \\ - K_{10}^i \dot{e}_z - K_{15}^i h_x - K_{16}^i h_z \end{aligned} \tag{41}$$

where $i = x,z$. Then, from the definition of ξ_ϕ and ξ_z, Eq. (40) becomes

$$\begin{aligned} \chi_i^{(\mathrm{V})} + 5\omega_0^2 \chi_i^{(\mathrm{III})} + 4\omega_0^4 \dot{\chi}_i &= K_4^i e_z + (K_5^i - 4\omega_0^4 K_9^i)\dot{e}_z \\ &+ (K_6^i - 4\omega_0^4 K_{10}^i)\ddot{e}_z + (K_7^i - 5\omega_0^2 K_9^i) e_z^{(\mathrm{III})} \\ &+ (K_8^i - 5\omega_0^2 K_{10}^i) e_z^{(\mathrm{IV})} + (K_{11}^i - 4\omega_0^4 K_{15}^i)\dot{h}_x \\ &+ K_{12}^i \ddot{h}_x + (K_{13}^i - 5\omega_0^2 K_{15}^i) h_x^{(\mathrm{III})} + K_{14}^i h_x^{(\mathrm{IV})} \end{aligned} \tag{42}$$

where $i = x,z$. Introduce variables ζ_x, ζ_z, η_x, and η_z such that

$$\ddot{\zeta}_x + \omega_0^2 \zeta_x = h_x \tag{43a}$$

$$\ddot{\eta}_x + 4\omega_0^2 \eta_x = h_x \tag{43b}$$

$$\ddot{\zeta}_z + \omega_0^2 \zeta_z = \psi - \psi_c \tag{43c}$$

$$\ddot{\eta}_z + 4\omega_0^2 \eta_z = \psi - \psi_c \tag{43d}$$

Then χ_i can be represented in terms of ζ and η as

$$\begin{aligned} \chi_i = \mathfrak{A}_i \zeta_z + \mathfrak{B}_i \dot{\zeta}_z + \mathfrak{C}_i \eta_z + \mathfrak{D}_i \dot{\eta}_z \\ + \mathfrak{E}_i \zeta_x + \mathfrak{F}_i \dot{\zeta}_x + \mathfrak{G}_i \eta_x + \mathfrak{H}_i \dot{\eta}_x, \qquad i = x,z \end{aligned} \tag{44}$$

where

$$\mathfrak{A}_i = ⅓(\omega_0^2 K_9^i - K_7^i + \omega_0^{-2} K_5^i)$$

$$\mathfrak{B}_i = ⅓(\omega_0^2 K_{10}^i - K_8^i + \omega_0^{-2} K_6^i - \omega_0^{-4} K_4^i)$$

$$\mathfrak{C}_i = -⅓(16\omega_0^2 K_9^i - 4K_7^i + \omega_0^{-2} K_5^y)$$

$$\mathfrak{D}_i = -(1/12)(64\omega_0^2 K_{10}^i - 16K_8^i + 4\omega_0^{-2} K_6^i - \omega_0^{-4} K_4^i)$$

$$\mathfrak{E}_i = ⅓(\omega_0^2 K_{15}^i - K_{13}^i + \omega_0^{-2} K_{11}^i)$$

$$\mathfrak{F}_i = -⅓(K_{14}^i - \omega_0^{-2} K_{12}^i)$$

$$\mathfrak{G}_i = -⅓(16\omega_0^2 K_{15}^i - 4K_{13}^i + \omega_0^{-2} K_{11}^i)$$

$$\mathfrak{H}_i = ⅓(4K_{14}^i - \omega_0^{-2} K_{12}^i)$$

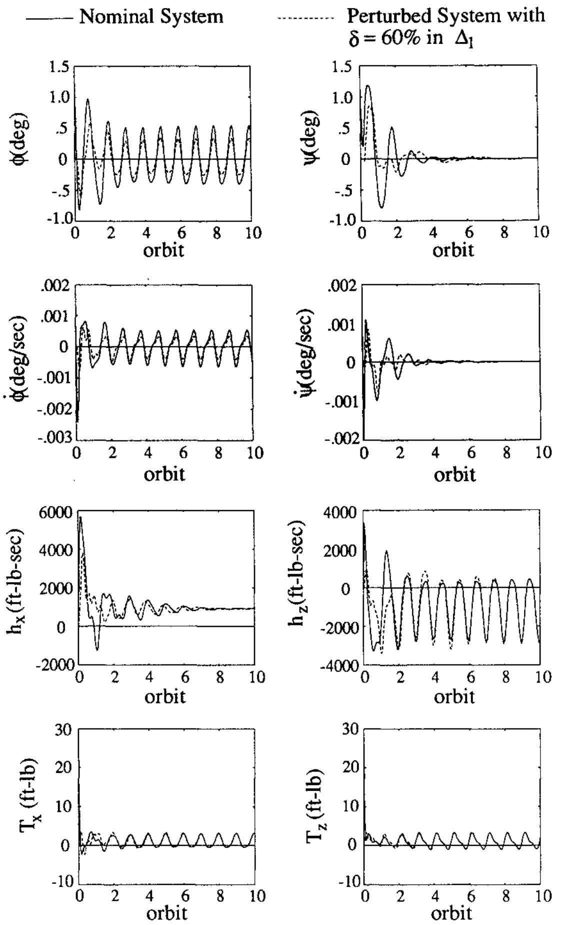


Fig. 5 Time response for the roll-yaw axis.

Combining Eqs. (41) and (44) yields an implementable form for the control torque T_i as

$$
\begin{aligned}
T_i = {} & \bar{K}_1^i \int (\phi - \phi_c)\,\mathrm{d}t + \bar{K}_2^i(\phi - \phi_c) + \bar{K}_3^i\dot{\phi} \\
& + \bar{K}_4^i \int (\psi - \psi_c)\,\mathrm{d}t + \bar{K}_5^i(\psi - \psi_c) + \bar{K}_6^i\dot{\psi} \\
& + \bar{K}_7^i h_x + \bar{K}_8^i h_z + \bar{K}_9^i \zeta_z + \bar{K}_{10}^i \dot{\zeta}_z + \bar{K}_{11}^i \eta_z \\
& + \bar{K}_{12}^i \dot{\eta}_z + \bar{K}_{13}^i \zeta_x + \bar{K}_{14}^i \dot{\zeta}_x + \bar{K}_{15}^i \eta_x + \bar{K}_{16}^i \dot{\eta}_x
\end{aligned}
$$

where $i = x,z$ and where

$$
\bar{K}_1^i = I_{in}K_1^i, \quad \bar{K}_2^i = I_{in}K_2^i, \quad \bar{K}_3^i = I_{in}K_3^i, \quad \bar{K}_4^i = \frac{I_{in}K_4^i}{4\omega_0^4}
$$

$$
\bar{K}_5^i = I_{in}K_9^i, \quad \bar{K}_6^i = I_{in}K_{10}^i, \quad \bar{K}_7^i = I_{in}K_{15}^i, \quad \bar{K}_8^i = I_{in}K_{16}^i
$$

$$
\bar{K}_9^i = I_{in}\mathcal{A}_i, \quad \bar{K}_{10}^i = I_{in}\mathcal{B}_i, \quad \bar{K}_{11}^i = I_{in}\mathcal{C}_i, \quad \bar{K}_{12}^i = I_{in}\mathcal{D}_i
$$

$$
\bar{K}_{13}^i = I_{in}\mathcal{E}_i, \quad \bar{K}_{14}^i = I_{in}\mathcal{F}_i, \quad \bar{K}_{15}^i = I_{in}\mathcal{G}_i, \quad \bar{K}_{16}^i = I_{in}\mathcal{H}_i
$$

The control torque described by Eq. (45) forms a dynamical feedback control law that consists of a conventional PID control and cyclic disturbance rejection filters. The integral feedbacks in Eq. (45) are expected to reject the constant input disturbance in attitude. Equation (43) represents the cyclic disturbance rejection filter for the yaw attitude and the roll CMG momentum. The initial states of the integrators in Eq. (45) and the cyclic disturbance rejection filter are the designer's choice.

Table 6 Controller gains and closed-loop eigenvalues in the roll-yaw channel

i	$\bar{K}_i^x$	$\bar{K}_i^z$	Unit
1	8.0779E−2	6.9890E−2	ft-lb/rad-s
2	9.2311E+2	7.5497E+2	ft-lb-rad
3	4.1496E+5	2.1476E+5	ft-lb-s/rad
4	−1.6675E−2	1.2840E−2	ft-lb/rad-s
5	−1.7214E+2	2.1965E+2	ft-lb/rad
6	1.6537E+5	3.6144E+5	ft-lb-s/rad
7	2.7958E−3	2.1891E−3	ft-lb/ft-lb-s
8	1.2508E−3	1.6052E−3	ft-lb/ft-lb-s
9	−1.0291E−4	−1.3627E−4	ft-lb-rad/s^2
10	−6.9590E−2	−7.9080E−2	ft-lb-rad/s
11	2.4992E−4	−3.3459E−4	ft-lb-rad/s^2
12	5.8883E−2	−2.4604E−2	ft-lb-rad/s
13	−1.1076E−10	−1.6908E−10	ft-lb/ft-lb-s^3
14	3.0504E−7	1.4664E−7	ft-lb/ft-lb-s^2
15	−1.0124E−9	−1.3267E−9	ft-lb/ft-lb-s^3
16	−4.6107E−7	−4.3079E−7	ft-lb/ft-lb-s^2

Closed-loop eigenvalues in roll-yaw channel	
−2.62, −1.75,	−1.05 ± 0.06j
−0.10 ± 0.98j	−0.28 ± 1.09j
−0.21 ± 0.06j	−0.23 ± 0.90j
−0.10 ± 1.97j	−0.38 ± 2.05j

For the roll-yaw channel controller design, ρ, γ, n_1, n_2, C_1, and C are chosen as

$$
\rho = 0.095, \qquad \gamma = 0.172, \qquad n_1 = n_2 = 5
$$

$$
C = \begin{bmatrix} \Omega_1 & 0_{10\times 6} \\ 0_{6\times 10} & \Omega_2 \end{bmatrix}, \qquad C_1 = 0_{2\times 2}
$$

$$
\Omega_1 = 4.6 \cdot \mathrm{diag}(2.4\omega_0^3 \;\, 0.1\omega_0^2 \;\, 0.1\omega_0 \;\, 2.4\omega_0^7 \;\, \omega_0^6 \;\, \omega_0^5 \;\, \omega_0^4 \;\, \omega_0^3 \;\, \omega_0^2 \;\, \omega_0)
$$

$$
\Omega_2 = 1.5 \cdot \mathrm{diag}\left(\frac{1}{I_{xn}}\omega_0^5 \;\, \frac{1}{I_{xn}}\omega_0^4 \;\, \frac{1}{I_{xn}}\omega_0^3 \;\, \frac{1}{I_{xn}}\omega_0^2 \;\, \frac{1}{I_{xn}}\omega_0 \;\, \frac{0.1}{I_{zn}}\omega_0\right)
$$

and the minimal non-negative definite solution to the corresponding ARE are taken. The controller gain matrix $\bar{K}$ and the closed-loop eigenvalues for roll-yaw channel are shown in Table 6. The largest closed-loop eigenvalues are seen to remain close to the orbital frequency. The stability margins in some specific variations are listed in Table 3. For all types of variations listed in Table 3 except Δ_1, Δ_3, and Δ_6, the designed controller stabilizes the system far beyond the physical limit, which means that good performance robustness is achieved for these directional variations. For Δ_1, Δ_3, and Δ_6, 62% stability margin is achieved. A simulation is performed with the parameter set considered in Ref. 2

$$
A_{1x}^d = A_{1z}^d = 1 \text{ ft-lb}, \quad A_{2x}^d = A_{2z}^d = 0.5 \text{ ft-lb}
$$

$$
B_x^d = B_z^d = 1 \text{ ft-lb}, \quad \varphi_{1x} = \varphi_{2x} = \varphi_{1z} = \varphi_{2z} = 0
$$

$$
\phi(0) = \psi(0) = 1 \text{ deg}, \quad \dot{\phi}(0) = \dot{\psi}(0) = 0.001 \text{ deg/s}
$$

$$
\phi_c = \psi_c = 0 \text{ deg}
$$

and the other initial conditions are zero. Figure 5 shows the time responses for the nominal system and a perturbed system with $\delta =$ 60% of the Δ_1 variation denoted by solid line and dotted line, respectively. With no noticeable performance degradation, the system appears to have good performance robustness. The constant disturbance torques are rejected in roll-yaw attitude channels while the cyclic disturbance torques are rejected in roll-CMG and yaw-attitude channels. The CMG momentum in the roll channel approaches a constant value while the CMG momentum in the yaw channel oscillates around a constant value. The biased CMG momentum in steady state can be changed by changing the command attitudes ϕ_c and ψ_c. The CMG momentum in roll-yaw channel is unbiased when the command attitudes are set to the TEA.

IV. Conclusions

The game theoretic controller is applied to momentum management and attitude control of the Space Station in the presence of uncertainty in the moments of inertia. The game theoretic controller has been developed for an uncertain linear time-invariant system by representing the uncertain dynamic system as an internal feedback loop and considering the input and the fictitious disturbance caused by parameter uncertainty as two noncooperative players. It was shown that this controller stabilizes the system for the prescribed parameter uncertainty bounds. Inclusion of the external disturbance torque to the design procedure results in a dynamical feedback controller that consists of conventional PID control and the cyclic disturbance rejection filter. This shows the state-space formulation for design provides a proper mechanization for handling the external disturbance. It was shown that the game theoretic design achieves a stability robustness with respect to inertia variations without sacrificing performance robustness and without increasing the system bandwidth.

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